Existence of the Schmidt decomposition for tripartite systems

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Abstract

For any bipartite quantum system the Schmidt decomposition allows us to express the state vector in terms of a single sum instead of double sums. We show the existence of the Schmidt decomposition for tripartite system under certain condition. If the partial inner product of a basis (belonging to a Hilbert space of smaller dimension) with the state of the composite system gives a disentangled basis, then the Schmidt decomposition for a tripartite system exists. In this case the reduced density matrix of each of the subsystem has equal spectrum in the Schmidt basis.

The key to quantum information processing is the entangled nature of quantum states, which has no classical counter part in the theory. These states are central to the study of quantum non-locality, quantum teleportation, quantum cryptography, dense coding and so on. In the simplest term entangled state is a one which cannot be expressed as a direct product of the states of two or more subsystems. If we have a quantum system consisting of two subsystems A and B, then the state of the combined system in general can be expressed as $|\psi\rangle_{AB} = \sum_{ij} a_{ij} |u_i\rangle \otimes |v_j\rangle$, where $\{|u_i\rangle\}, (i = 1, 2, ...N_A) \in \mathcal{H}_A$ and $\{|v_i\rangle\}, (i = 1, 2, ...N_B) \in \mathcal{H}_B$ are the complete set of orthonormal basis vectors in their respective Hilbert spaces. The expansion coefficients in the above state of the combined system contain $N_A N_B$ terms and is very difficult to manipulate with. However, the Schmidt decomposition (SD) theorem [1] comes to rescue us. It says that any arbitrary state of a bipartite (two-subsystem) quantum system can be expressed as [2]

$$|\psi\rangle_{AB} = \sum_{i=1}^{N_A} \sqrt{p_i} |x_i\rangle_A \otimes |y_i\rangle_B, \tag{1}$$

where $\{|x_i\rangle_A\}$ and $\{|y_i\rangle_B\}$ are two orthonormal basis sets belonging to Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively and $N_A \leq N_B$. The expansion coefficients $\sqrt{p_i}$ can be chosen to be real and positive. This simplifies the expression to a great extent. The Schmidt decomposition theorem has been applied in many worlds interpretation of quantum theory [3], in proving Bell's inequality [4,5] and is quite successful. Recently, in quantum optics context this has

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been applied [6] and a geometric approach [8] to Schmidt decomposition of two-spin of particles is given in relation to Hardy's proof of quantum non-locality. However, all these discussions pertains strictly to bipartite systems only.

If we have a composite system consisting of more than two subsystems there does not exist a Schmidt decomposition in general. But if it would exist, it will be quite useful simply because the number of terms one deals with in the expansion of the state vector will be comprehensibly small. For example, if we have a quantum system comprising of three subsystems (a tripartite system) then the general state of the system is given by

$$|\psi\rangle_{ABC} = \sum_{ijk} a_{ijk} |u_i\rangle_A \otimes |v_j\rangle_B \otimes |w_k\rangle_C, \tag{2}$$

where $\{|u_i\rangle\} \in \mathcal{H}_A = C^{N_A}$, $\{|v_j\rangle\} \in \mathcal{H}_B = C^{N_B}$ and $\{|w_k\rangle\} \in \mathcal{H}_C = C^{N_C}$. In this case there are $N_A N_B N_C$ terms to be dealt with. On the other hand if a Schmidt decomposition for tripartite system exists , then we can write the state in (2) as

$$|\psi\rangle_{ABC} = \sum_{i} \sqrt{p}_i |x_i\rangle_A \otimes |y_i\rangle_B \otimes |z_i\rangle_C, \tag{3}$$

where $i = 1, 2, ...N_A$ (say) if $N_A = \dim \mathcal{H}_A$ is the smallest of all the three and $\{|x_i\rangle_A\}$, $\{|y_i\rangle_B\}$ and $|w_i\rangle_C$ are again orthonormal basis sets belonging to their respective Hilbert spaces. The general argument which goes against the existence of Schmidt decomposition for a tripartite system such as (3) is that the "equal-spectrum" condition for reduced density matrices does not hold [7]. Nevertheless, it is worth exploring under what conditions SD can exist. If the Schmidt decomposition for a tripartite system exists it would be useful in modal interpretation of quantum theory [9–11], for example. This possibility has been explored by Peres [12] and he found a necessary and sufficient condition for the existence of a Schmidt decomposition for a tripartite system. However, his condition does not give insight why does it fail for some tripartite systems and why does it *always work* for a bipartite system. Recently, SD has been discussed for multipartite systems in connection with quantifying the entanglment [13]. In this paper we find a simple criterion for the existence of Schmidt decomposition for tripartite system. This gives insight to the question: why does it work always for a bipartite not for a tripartite system. To state it simply, we prove that if the partial inner product of a basis of any one of the subsystems (belonging to a Hilbert space of smaller dimension) with the state of the composite system gives a disentangled basis, then Schmidt decomposition for a tripartite system exists. If the partial inner product gives an entangled basis the Schmidt decomposition in terms of a single sum does not exist, though the triple sum can be converted to a double sum. Using our existence criterion we show that the reduced density matrix of each subsystem (by taking partial traces over any two subsystems) has the same eigen values, i.e. the "equal-spectrum" requirement holds. Our criterion is also consistent with the existence of Schmidt decomposition for a bipartite system. When we take the partial inner product of any one of the basis with the state of an arbitrary bipartite system, then the resulting basis belong to a Hilbert space of a single subsystem (no question of an entangled basis). This is the main reason why the Schmidt decomposition always works for a bipartite system.

Now we prove the following theorem.

Theorem: For any state $|\psi\rangle_{ABC} \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ of a tripartite system let $\dim \mathcal{H}_A = N_A$ is smallest of N_B and N_C. If the "partial inner product" of the basis $|u_i\rangle_A$ with the state $|\psi\rangle_{ABC}$, i.e. ${}_A\langle u_i|\psi\rangle_{ABC} = |\psi_i\rangle_{BC}$ has Schmidt number one then the Schmidt decomposition for a tripartite system exists.

Proof: Let $|\psi\rangle_{ABC} = \sum_{ijk} a_{ijk}|u_i\rangle_A \otimes |v_j\rangle_B \otimes |w_k\rangle_C$ and the partial inner product of the basis $|u_i\rangle$ and state $|\psi\rangle_{ABC}$ is a basis vector in the Hilbert space $\mathcal{H}_B \otimes \mathcal{H}_C$ spanned by basis vectors $\{|v_j\rangle_B \otimes |w_k\rangle_C\}$. It is given by

$$|\psi_i\rangle_{BC} = \sum_{jk} a_{ijk}|v_j\rangle_B \otimes |w_k\rangle_C, \tag{4}$$

where $\{|\psi_i\rangle_{BC}\}$ is an orthogonal basis set but need not be normalised. We know that any vector in a bipartite Hilbert space can be written as a Schmidt decomposition form, i.e.

$$|\psi_i\rangle_{BC} = \sum_{\mu} \sqrt{p}_{\mu}^{(i)}|y_\mu\rangle_B \otimes |z_\mu\rangle_C, \tag{5}$$

where $\{|y_\mu\rangle_B\}$ and $\{|z_\mu\rangle_C\}$ are orthonormal basis for $\mathcal{H}_B$ and $\mathcal{H}_C$ which can be unitarily related to the basis $\{|v_j\rangle_B\}$ and $\{|w_k\rangle_C\}$, respectively. Therefore, the arbitrary state in general can be written as

$$|\psi\rangle_{ABC} = \sum_{i\mu} \sqrt{p}_{\mu}^{(i)}|u_i\rangle_A \otimes |y_\mu\rangle_B \otimes |z_\mu\rangle_C, \tag{6}$$

Now there can be two situations: (i) either the basis (we call bi-Schmidt basis) $\{|\psi_i\rangle_{BC}\}$ is entangled or (ii) it is separable. We can apply the pure state entanglement criterion, i.e., if the Schmidt number is equal to one then the state is separable and if it is greater than one it is entangled. Here, the Schmidt number is nothing but the number of non-zero eigenvalues in the reduced density matrix of a bipartite system and is the same as the number of terms in the Schmidt decomposition of a bipartite state. This is a good measure of entanglement for pure states. Now imposing this condition on bi-Schmidt basis, we write it as a separable form. So, if $|\psi_i\rangle_{BC}$ has Schmidt number one we can write $|\psi_i\rangle_{BC} = |\beta_i\rangle_B \otimes |\gamma_i\rangle_C$. Since $\{|\psi_i\rangle_{BC}\}$ is not normalised $\{|\beta_i\rangle_B\}$ and $\{|\gamma_i\rangle_C\}$ need not be orthonormal though they satisfy orthogonality condition. Therefore, the tripartite system can be written as

$$|\psi\rangle_{ABC} = \sum_i |u_i\rangle_A \otimes |\beta_i\rangle_B \otimes |\gamma_i\rangle_C. \tag{7}$$

Now we calculate the reduced density matrix of each subsystem. The reduced density matrix for A can be obtained by taking partial traces over B and C. Thus,

$$\rho_A = \mathrm{tr}_B(\mathrm{tr}_C(\rho_{ABC})) = \mathrm{tr}_B[\sum_i q_i|u_i\rangle_{AA}\langle u_i| \otimes |\beta_i\rangle_{BB}\langle\beta_i|], \tag{8}$$

where we have used the trace equality $\mathrm{tr}_C(|\gamma_i\rangle_C\langle\gamma_j|) = {}_C\langle\gamma_j|\gamma_i\rangle_C = q_i\delta_{ij}$ and $q_i = \parallel \gamma_i \parallel^2$ is the (squared) norm of the basis vector $|\gamma_i\rangle_C$. Performing the second trace we can write the above one as

$$\rho_A = \sum_i q_i r_i |u_i\rangle_{AA}\langle u_i|, \tag{9}$$

where we have used $\mathrm{tr}_B(|\beta_i\rangle_B\langle\beta_j|) = {}_B\langle\beta_j|\beta_i\rangle_B = r_i\delta_{ij}$ and $r_i = \parallel \beta_i \parallel^2$ is the (squared) norm of the basis vector $|\beta_i\rangle$. Similarly, we can obtain the reduced density matrix ρ_B and ρ_C as

$$
\begin{aligned}
\rho_B &= \sum_i q_i r_i |\beta'_i\rangle_{BB}\langle\beta'_i| \\
\rho_C &= \sum_i q_i r_i |\gamma'_i\rangle_{CC}\langle\gamma'_i|,
\end{aligned}
\tag{10}
$$

where we have defined the orthonormal basis vectors $|\beta'_i\rangle_B$ and $|\gamma'_i\rangle_C$ for $\mathcal{H}_B$ and $\mathcal{H}_C$ as $|\beta_i\rangle_B = \sqrt{r}_i|\beta'_i\rangle_B$ and $|\gamma_i\rangle_C = \sqrt{q}_i|\gamma'_i\rangle_C$. By comparing all the density matrices, we see that they have same eigenvalue spectrum $\{q_i r_i\}$ in the Schmidt basis . Now we can redefine the state of the tripartite system as

$$
\begin{aligned}
|\psi\rangle_{ABC} &= \sum_i \sqrt{q_i r_i}|u_i\rangle_A \otimes |\beta'_i\rangle_B \otimes |\gamma'_i\rangle_C \\
&= \sum_i \sqrt{d}_i |i\rangle_A|i\rangle_B|i\rangle_C.
\end{aligned}
\tag{11}
$$

This is the Schmidt decomposition for a tripartite system and hence the proof.

It should be remarked that ρ_A, ρ_B and ρ_C have the same number of distinct non-zero eigenvalues (non-degenerate spectrum), however, the number of zero-eigen values of ρ_A, ρ_B and ρ_C can be different as $\mathcal{H}_A, \mathcal{H}_B$ and $\mathcal{H}_C$ have different dimensions. The Schmidt decomposition (11) is unique for non-degenerate spectrum of reduced density matrices. The same is true for a bipartite system. Further, when we have a bipartite system then the "partial inner product" of any of the basis with the state of the system gives a single (disentangled) basis for the other one. Hence, the SD is always posible for a bipartite system.

Once we know that the SD exists, then no local unitary operation of the form $U_B \otimes U_C$, local general measurements and classical communication can disprove the existence of Schmidt decomposition. We know that any measure of entanglement $E(\rho_{i,BC})$, with $\rho_{i,BC} = |\psi_i\rangle_{BCB}C\langle\psi_i|$ satisfies the requirements [14] (i) $E(\rho_{i,BC}) = 0$ when $\rho_{i,BC}$ is separable, (ii) $E(\rho_{i,BC}) = E(U_B \otimes U_C \rho_{i,BC} U_B^\dagger \otimes U_C^\dagger)$, and (iii) $E(\rho_{i,BC})$ cannot increase under local general measurement and classical communication, the Schmidt number of the bi-Schmidt basis does not change and it is impossible to disprove the existence of Schmidt decomposition.

Next we discuss the situation when Schmidt decomposition does not exist for a tripartite system. This is the case when the bi-Schmidt basis is an entangled basis. Physically, this means there exists "*entanglement within entanglement*". When bi-Schmidt basis is separable there is entanglement between each subsystem A, B and C and there is no " entanglement within entanglement". When there is "entanglement within entanglement" the "equal-spectrum" requirement breaks down. However, the "equal-spectrum" holds within the subsystems B and C, i.e., ρ_B and ρ_C have same non-zero eigenvalues. To see this consider the state of a tripartite system as $|\psi\rangle_{ABC} = \sum_i |u_i\rangle_A \otimes |\psi_i\rangle_{BC}$. The density matrix of the tripartite system is given by

$$
\rho_{ABC} = \sum_{ij} |u_i\rangle_{AA}\langle u_j| \otimes |\psi_i\rangle_{BCBC}\langle\psi_j|. \tag{12}
$$

On tracing over subsystem B and C we have the reduced density matrix ρ_A given by

$$
\rho_A = \sum_i p_i |u_i\rangle_{AA}\langle u_i|, \tag{13}
$$

where we have used the trace equality $\mathrm{tr}_{BC}(|\psi_i\rangle_{BC\,BC}\langle\psi_j|) = {}_{BC}\langle\psi_j|\psi_i\rangle_{BC} = p_i\delta_{ij}$ and ${}_{BC}\langle\psi_i|\psi_i\rangle_{BC} = \sum_k p_k^{(i)} = p_i$ is the (squared) norm of the bi-Schmidt basis. The reduced density matrix ρ_{AB} given by

$$\rho_{AB} = \mathrm{tr}_C(\rho_{ABC}) = \sum_{ij\mu} \sqrt{p_\mu^{(i)} p_\mu^{(j)}} |u_i\rangle_{AA}\langle u_j| \otimes |y_\mu\rangle_{BB}\langle y_\mu|. \quad (14)$$

On tracing over the subsystem A we get the reduced density matrix for ρ_B given by

$$\rho_B = \sum_{i\mu} p_\mu^{(i)} |y_\mu\rangle_B\langle y_\mu| = \sum_\mu s_\mu |y_\mu\rangle_{BB}\langle y_\mu|, \quad (15)$$

where we have defined $\sum_i p_\mu^{(i)} = s_\mu$ and each of them are some positive numbers. To obtain the reduced density matrix for C we first note that ρ_{AC} is given by

$$\rho_{AC} = \mathrm{tr}_B(\rho_{ABC}) = \sum_{ij\mu} \sqrt{p_\mu^{(i)} p_\mu^{(j)}} |u_i\rangle_{AA}\langle u_j| \otimes |z_\mu\rangle_{CC}\langle z_\mu|. \quad (16)$$

From the above one we get the reduced density matrix for ρ_C given by

$$\rho_C = \sum_{i\mu} p_\mu^{(i)} |z_\mu\rangle_{CC}\langle z_\mu| = \sum_\mu s_\mu |z_\mu\rangle_{CC}\langle z_\mu|. \quad (17)$$

From (15) and (17) these we can see that ρ_B and ρ_C have same eigenvalue spectrum $\{s_\mu\}$ whereas the eigenvalue of ρ_A has different spectrum $\{p_i\}$. Thus, the "equal-spectrum" requirement breaks down when the bi-Schmidt basis $\{|\psi_i\rangle_{BC}\}$ has Schmidt number greater than one. Interestingly, if we look the subsystems B and C as a single subsystem BC, then the "equal-spectrum" requirement holds for subsystems A and BC. This is because we have

$$\rho_{BC} = \sum_i |\psi_i\rangle_{BC\,BC}\langle\psi_i|. \quad (18)$$

On defining an orthonormal basis $|\psi_i'\rangle_{BC}$ as $|\psi_i\rangle_{BC} = \sqrt{p}_i |\psi_i'\rangle_{BC}$, we have

$$\rho_{BC} = \sum_i p_i |\psi_i'\rangle_{BC\,BC}\langle\psi_i'| \quad (19)$$

This shows that ρ_A and ρ_{BC} have equal-spectrum as expected intuitively.

In conclusion, we have found a simple criterion for the existence of Schmidt decomposition for tripartite system and discussed why does it fail in some cases. This also answers why does it always works for a bipartite system. The existence of Schmidt decomposition might be useful in quantifying entanglement content of a pure tripartite system. For example, if the SD exists then the von Neumann entropy of any of the reduced density matrix would give the entanglement content of a pure tripartite system. It would be very interesting to see if one can say more about the SD for multipartite entangled systems.

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